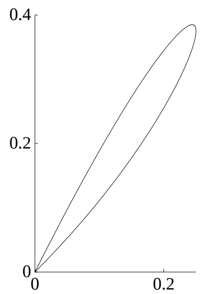
# Green's Theorem

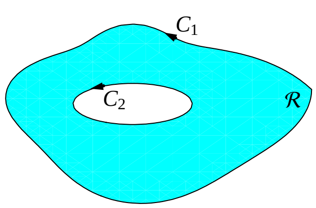
1. Let $C$ be the boundary of the unit square $0 \le x \le 1, 0 \le y \le 1$, oriented counterclockwise, and let $\vec{F}$ be the vector field $\vec{F}(x,y) = \langle e^y + x, x^2 - y \rangle$. Find $\displaystyle\int_C \vec{F} \cdot d\vec{r}$.

2. Let $C$ be the oriented curve consisting of line segments from $(0,0)$ to $(2,3)$ to $(2,0)$ back to $(0,0)$, and let $\vec{F}(x,y) = \langle y^2, x^2 \rangle$. Find $\displaystyle\int_C \vec{F} \cdot d\vec{r}$.

3. Find the area of the region enclosed by the parameterized curve $\vec{r}(t) = \langle t - t^2, t - t^3 \rangle$, $0 \le t \le 1$.




4. Let $\vec{F}(x,y) = \langle P(x,y), Q(x,y) \rangle$ be any vector field defined on the region $\mathcal{R}$ (in $\mathbb{R}^2$) shown in the picture, and let $C_1$ and $C_2$ be the oriented curves shown in the picture. What does Green's Theorem say about $\displaystyle\int_{C_1} \vec{F} \cdot d\vec{r}$, $\displaystyle\int_{C_2} \vec{F} \cdot d\vec{r}$, and $\displaystyle\iint_{\mathcal{R}} (Q_x - P_y)\, dA$?



5. Let $\vec{F}(x,y) = \langle P(x,y), Q(x,y) \rangle = \left\langle \dfrac{x}{\sqrt{x^2+y^2}}, \dfrac{y}{\sqrt{x^2+y^2}} \right\rangle$. You can check that $P_y = Q_x$.

(a) What is wrong with the following reasoning? "$P_y = Q_x$, so $\vec{F}$ is conservative."

*In the remainder of this problem, you will show that $\vec{F}$ is conservative by showing that $\vec{F}$ satisfies the closed loop property. (That is, if $C$ is any closed curve, then $\int_C \vec{F} \cdot d\vec{r} = 0$.) We observed last time that this seemed like an impossible task; now that we know Green's Theorem, it's much more doable.*

(b) Let $C$ be any simple closed curve in $\mathbb{R}^2$ that does *not* enclose the origin, oriented counterclockwise. (A simple curve is a curve that does not cross itself.) Use Green's Theorem to explain why $\int_C \vec{F} \cdot d\vec{r} = 0$.

(c) Let $a$ be a positive constant, and let $C$ be the circle $x^2 + y^2 = a^2$, oriented counterclockwise. Parameterize $C$ (check your parameterization by plugging it into the equation $x^2 + y^2 = a^2$), and use the definition of the line integral to show that $\int_C \vec{F} \cdot d\vec{r} = 0$. (Why doesn't the reasoning from (b) work in this case?)

(d) Let $C$ be any simple closed curve in $\mathbb{R}^2$ that *does* enclose the origin, oriented counterclockwise. Explain why $\int_C \vec{F} \cdot d\vec{r} = 0$. (Hint: Use (c) and #4.)

(e) Is it valid to conclude from the above reasoning that, if $\vec{F}(x, y) = \langle P(x, y), Q(x, y)\rangle$ is a vector field defined everywhere except the origin and $P_y = Q_x$, then $\vec{F}$ is conservative?

6. In this problem, you'll prove Green's Theorem in the case where the region is a rectangle. Let $\vec{F}(x, y) = \langle P(x, y), Q(x, y)\rangle$ be a vector field on the rectangle $\mathcal{R} = [a, b] \times [c, d]$ in $\mathbb{R}^2$.

   (a) Show that $\displaystyle\iint_{\mathcal{R}} [Q_x(x, y) - P_y(x, y)]\ dA = \int_c^d [Q(b, y) - Q(a, y)]\ dy - \int_a^b [P(x, d) - P(x, c)]\ dx.$

   (b) Let $C$ be the boundary of $\mathcal{R}$, traversed counterclockwise. Show that $\displaystyle\int_C \vec{F} \cdot d\vec{r}$ is also equal to $\displaystyle\int_c^d [Q(b, y) - Q(a, y)]\ dy - \int_a^b [P(x, d) - P(x, c)]\ dx.$